MATH3705 Tutorial 4

1. Let
$$f(x) = \begin{cases} 2, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in (0, \pi). \end{cases},$$
and let $f(x)$ be 2π-periodic. Find the Fourier series of $f(x)$ and determine the sums to which the series converges at $x = 0, 101\pi, 88.1\pi$.

Solution:

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2dx + \int_{0}^{\pi} 1dx\right) = 3, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2\cos(nx)dx + \int_{0}^{\pi} 1\cos(nx)dx\right) = 0, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2\sin(nx)dx + \int_{0}^{\pi} 1\sin(nx)dx\right) \\
&= -\frac{2}{n\pi}\cos(nx)|_{-\pi}^{0} - \frac{1}{n\pi}\cos(nx)|_{0}^{\pi} = \frac{1}{n\pi}\left(-1+(-1)^n\right) = \begin{cases} -\frac{2}{n\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases}
\end{aligned}$$

Hence the Fourier series for $f(x)$ is:

$$\begin{aligned}
f(x) &\sim 1.5 - \sum_{odd\ n} \frac{2}{n\pi}\sin(nx) = \\
&= 1.5 - \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi}\sin(2n+1)x, \quad \forall x \in (-\pi, \pi).
\end{aligned}$$

$f_{av}(0) = (2+1)/2 = 1.5$, $f_{av}(101\pi) = f_{av}(50(2\pi)+\pi) = f_{av}(\pi) = (2+1)/2 = 1.5$, $f_{av}(88.1\pi) = f_{av}(44(2\pi)+0.1\pi) = f_{av}(0.1\pi) = 1$.

2. Let $f(x) = 2x$ on [0, 2].

(a) Find the Fourier series of $f(x)$ and determine the values to which the series converges at x = 47, 47.5, 49 and 50.

Solution: We extend $f(x)$ to be 2-periodic. $L = 1$. Discontinuous points are $0, \pm 2, \pm 4, \ldots.$

$$\begin{aligned} a_0 &= \frac{1}{1}\int_0^2 f(x)dx = 4, \\ a_n &= \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)dx = \int_0^2 2x\cos(n\pi x)dx = 0, \\ b_n &= \frac{1}{1}\int_0^2 f(x)\sin(n\pi x)dx = \int_0^2 2x\sin(n\pi x)dx = -\frac{4}{n\pi}. \end{aligned}$$

Thus

$$f(x) = 2 - \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin(n\pi x).$$

At x = 47, the series converges to 2.

At x = 47.5, the series converges to 3.

At x = 49, the series converges to 2.

At x = 50, the series converges to 2.

(b) Find the Fourier cosine series of $f(x)$ and determine the values to which the series converges at x = 47, 47.5, 49 and 50.

Solution: We extend $f(x)$ to be 4-periodic even function.

$$f_{even}(x) = \begin{cases} f(x), & x \in (0,2); \\ f(-x), & x \in (-2,0). \end{cases} = \begin{cases} 2x, & x \in (0,2); \\ -2x, & x \in (-2,0). \end{cases}$$

Discontinuous points are $\pm 2, \pm 6, \pm 10, \ldots.$

$$\begin{aligned} a_0 &= \frac{2}{2}\int_0^2 f(x)dx = 4, \\ a_n &= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \frac{8}{n^2\pi^2}[(-1)^n - 1], \\ b_n &= \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx = 0. \end{aligned}$$

Thus

$$f(x) = 2 + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi x}{2}.$$

At x = 47, the series converges to 2.

At x = 47.5, the series converges to 1.

At x = 49, the series converges to 2.

At x = 50, the series converges to 4.

(c) Find the Fourier sine series of $f(x)$ and determine the values to which the series converges at x = 47, 47.5, 49 and 50.

Solution: We extend $f(x)$ to be 4-periodic odd function.

$$f_{even}(x) = \begin{cases} f(x), & x \in (0,2); \\ -f(-x), & x \in (-2,0). \end{cases} = \begin{cases} 2x, & x \in (0,2); \\ 2x, & x \in (-2,0). \end{cases}$$

Discontinuous points are $\pm 2, \pm 6, \pm 10, \ldots.$

$$b_n = \frac{2}{2}\int_0^2 f(x) \sin \frac{n\pi x}{2} dx = \frac{8(-1)^{n+1}}{n\pi},$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{2}.$$

At x = 47, the series converges to -2.

At x = 47.5, the series converges to -1.

At x = 49, the series converges to 2.

At x = 50, the series converges to 0.

3. Let $f(x) = \begin{cases} x, & 0 \le x \le 1; \\ even, & \\ 2-periodic. & \end{cases}$

Find a 2-periodic solution of the equation

$$y'' + 2y = f(x), \quad -\infty < x < \infty.$$

Solution: Here $L = 1$, $\lambda = 2$. Since f is even, $b_n = 0$ for $n \ge 1$.

$$a_0 = 2\int_0^1 x dx = 1.$$

$$a_n = 2\int_0^1 x \cos(n\pi x) dx = \frac{2[(-1)^n - 1]}{n^2\pi^2}, \quad n \ge 1.$$

This implies that $c_0 = \frac{a_0}{\lambda} = \frac{1}{2}$, $c_n = \frac{a_n}{2-n^2\pi^2} = \frac{2[(-1)^n-1]}{n^2\pi^2(2-n^2\pi^2)}$ for all $n \ge 1$ and

$$d_n = 0.$$

$$y = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi^2(2 - n^2\pi^2)} \cos(n\pi x).$$